Proof of the Superposition Principle Differential Equations X. Du

- Proof of the Superposition Principle:
  - Let y_1 and y_2 be solutions to a linear homogeneous differential equation.
  - Lemma 1. cy_1 is a solution to the differential equation.
  - Lemma 2. $y_1 + y_2$ is also a solution to the differential equation.
  - Thus $y = c_1 y_1 + c_2 y_2$ is also a solution to the differential equation.
  - Suppose that $y_1,\ y_2, \ldots, y_n$ are all solutions of a linear homogeneous differential equation. Then, any linear combination of these solutions is also a solution (this result is trivial based on what we previously proved).
    - i.e. $y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is a solution
- Proof of Lemma 1:
  - cy_1 is a solution to $y^{(n)} + A(x)y^{(n-1)} + \ldots + P(x)y' + Q(x)y = 0$ iff $(cy_1)^{(n)} + A(x)(cy_1)^{(n-1)} + \ldots + P(x)(cy_1)' + Q(x)cy_1 = 0$ is true.
  - Given that y_1 is a solution to $y^{(n)} + A(x)y^{(n-1)} + \ldots + P(x)y' + Q(x)y = 0$, then $y_1^{(n)} + A(x)y_1^{(n-1)} + \ldots + P(x)y_1' + Q(x)y_1 = 0$ must be true.
  - $cy_1^{(n)} + A(x)cy_1^{(n-1)} + \ldots + P(x)cy_1' + Q(x)cy_1 = 0$ (multiply by c).
  - $(cy_1)^{(n)} + A(x)(cy_1)^{(n-1)} + \ldots + P(x)(cy_1)' + Q(x)cy_1 = 0$. Apply linearity of derivatives.
  - $\therefore cy_1$ is a solution to $y^{(n)} + A(x)y^{(n-1)} + \ldots + P(x)y' + Q(x)y = 0$
- Proof of Lemma 2:
  - $y_1 + y_2$ is a solution to $y^{(n)} + A(x)y^{(n-1)} + \ldots + P(x)y' + Q(x)y = 0$ iff $(y_1 + y_2)^{(n)} + A(x)(y_1 + y_2)^{(n-1)} + \ldots + P(x)(y_1 + y_2)' + Q(x)(y_1 + y_2) = 0$ is true.
  - Given that y_1 and y_2 are solutions to $y^{(n)} + A(x)y^{(n-1)} + \ldots + P(x)y' + Q(x)y = 0$, then $y_1^{(n)} + A(x)y_1^{(n-1)} + \ldots + P(x)y_1' + Q(x)y_1 = 0$ and $y_2^{(n)} + A(x)y_2^{(n-1)} + \ldots + P(x)y_2' + Q(x)y_2 = 0$ must be true.
  - $\left(y_1^{(n)} + A(x)y_1^{(n-1)} + \ldots + P(x)y_1' + Q(x)y_1\right) + \left(y_2^{(n)} + A(x)y_2^{(n-1)} + \ldots + P(x)y_2' + Q(x)y_2\right) = 0$
  - $y_1^{(n)} + y_2^{(n)} + A(x)(y_1^{(n-1)} + y_2^{(n-1)}) + P(x)(y_1' + y_2') + Q(x)(y_1 + y_2) = 0$
  - $(y_1 + y_2)^{(n)} + A(x)(y_1 + y_2)^{(n-1)} + \ldots + P(x)(y_1 + y_2)' + Q(x)(y_1 + y_2) = 0$. Apply linearity of derivatives.
  - $\therefore y_1 + y_2$ is a solution to $y^{(n)} + A(x)y^{(n-1)} + \ldots + P(x)y' + Q(x)y = 0$